Atomic levels in superstrong magnetic fields and $D = 2$ QED of massive electrons: screening

M.I.Vysotsky

ITEP, Moscow

Galitskii School of Theoretical Physics

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introduction

A.I.Vainstein (V.M.G. diploma student in early 60'):
"Galitskii was a great expert in Quantum Mechanics"

V.M.Galitskii, B.M. Karnakov, V.I.Kogan, Problems in Quantum Mechanics, 1992, problem 8.61:
...Ground level of hydrogen atom in strong B...
May be the longest solution (8 pages).

The same problem can be found in L.D.Landau, E.M.Lifshitz Quantum Mechanics , editions after 1974.

plan

- $a_B, a_H, \ a_H << a_B \Longrightarrow B >> e^3 m_e^2$
  electrons on Landau levels feel weak Coulomb potential moving along axis z;
  Loudon, Elliott 1960, numerical solution of Schrodinger equation;
  LL, GKK: $E_0 = -(me^4/2) \times \ln^2(B/(m^2 e^3))$. Ground level goes to $-\infty$ when B goes to ∞

  NO

- $D = 2$ QED - Schwinger model with massive electrons, radiative "corrections" to Coulomb potential in $d = 1$;
  $\Pi_{\mu\nu}$, interpolating formula, analytical formula for $\Phi(z)$,
  $g > m$ - photon "mass" $m_\gamma \sim g$, screening at ALL z when $g > m$

- $D = 4$ QED; photon “mass” $m_\gamma^2 = e^3 B$ at superstrong magnetic fields $B >> m_e^2/e^3 = 137 \times 4.4 \times 10^{13}$ gauss; asymptotic behaviour of $\Phi(z)$ at $z >> 1/m_e$ (no screening) and at $z << 1/m_e$ (photon “mass” and screening)
- ground state hydrogen atom energy in the superstrong magnetic field; excited levels
- References
- Conclusions

hydrogen atom in strong B

$$d = 3 : (p^2/(2m) - e^2/r)\chi(r) = E\chi(r)$$

$$R(r) = \chi(r)/r, r \geq 0, \chi(0) = 0$$

$$(\chi(0) \neq 0 \;\; \triangle 1/r = \delta(r))$$

$$d = 1 : (p^2/(2m) - e^2/|z|)\Psi(z) = E\Psi(z)$$

$$-\infty < z < \infty, \;\; \Psi(0) \neq 0$$

variational method for ground state energy:

$$\Psi(z) \sim exp(-|z|/b);$$

$$< V > \sim \ln(1/\epsilon)$$

$d = 1 \Longrightarrow d = 3$ at $z < a_H \equiv 1/\sqrt{eB}$ - Landau radius

$$V(z) = 1/\sqrt{z^2 + a_H^2}$$

$$\ln(1/\epsilon) \Longrightarrow 2\ln(a_B/a_H) = \ln(B/(m^2 e^3))$$

($a_B = 1/(me^2)$ - Bohr radius)

$$E_0 = -2m\left(\int_{a_H}^{a_B} U(z)dz\right)^2 \qquad (1)$$

$$E_0 = -(me^4/2) \times \ln^2(B/(m^2 e^3))$$

LL, GKK, BUT: Elliott, Loudon - numerical solution of d=1 Schrodinger equation...

first excited level: $\Psi_1(0) = 0, E_1 \Longrightarrow -me^4/2$ $(B \Longrightarrow \infty)$; degeneracy of odd and even levels; the only nondegenerate level - $E_0 \Longrightarrow -\infty$. One-dimensional Coulomb problem - Loudon (1959).

B values

Definitions (for this talk): $B > m_e^2 e^3 = 2.4 * 10^9$Gauss - strong B, $B > m_e^2/e^3 = 6 * 10^{15}$Gauss - superstrong B.

$B_{cr} = m_e^2/e = 4.4 * 10^{13}$Gauss - critical B

B in laboratories: $2 * 10^5$Gauss - CMS, Atlas; $10^6 - 10^7$Gauss - magnetic cumulation, A.D.Saharov, 1952, $H * r^2 =$const

Pulsars: $B \sim 10^{13}$Gauss; Magnetars: $B \sim 10^{15}$Gauss

Elliott, Loudon: excitons in semiconductors, $m \ll m_e$

superstrong B

QED loop corrections to photon propagator drastically change E_0 for $B >> m_e^2/e^3$.
Dirac equation spectrum in a constant homogenious magnetic field looks like:

$$\varepsilon_n^2 = m^2 + p_z^2 + (2n+1)eB + \sigma eB \quad , \tag{2}$$

where $n = 0, 1, 2, ...$, $\sigma = \pm 1$ (Rabi, 1928, $2n + 1 + \sigma \Longrightarrow 2j, \ j = 0, 1, 2, ...$)
$\varepsilon_n \gtrsim m/e$ - ultrarelativistic electrons; the only exception is the lowest Landau level (LLL) which has $n = 0$, $\sigma = -1$.
We will study states on which LLL splits in the field of nucleus.
Hydrogen atom: electron on LLL moves along axis z; proton stay at $z = 0$. What electric potential does electron feel? Let us look at $D = 2, d = D - 1 = 1$ QED.

$D = 2$ QED: screening of Φ

$$\Phi(\bar{k}) \equiv A_0(\bar{k}) = \frac{4\pi g}{\bar{k}^2}\ ; \quad \mathbf{\Phi} \equiv \mathbf{A}_0 = D_{00} + D_{00}\Pi_{00}D_{00} + \ldots$$

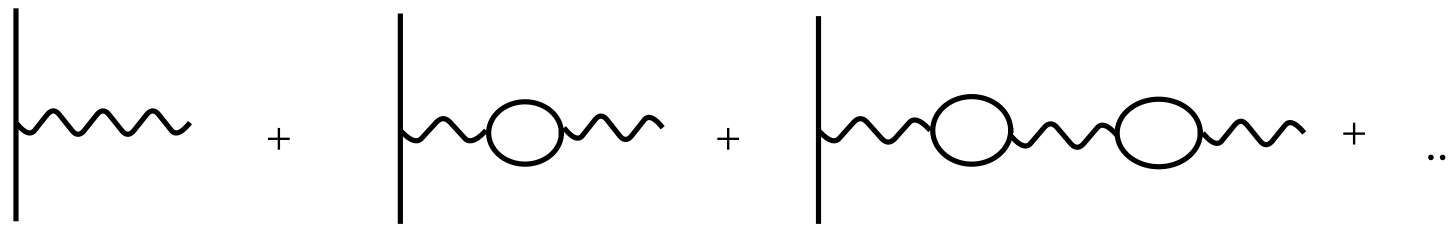


Fig 1. *Modification of the Coulomb potential due to the dressing of the photon propagator.*

Summing the series we get:

$$\mathbf{\Phi}(k) = -\frac{4\pi g}{k^2 + \Pi(k^2)}\ , \quad \Pi_{\mu\nu} \equiv \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\Pi(k^2) \qquad (3)$$

$$\Pi(k^2) = 4g^2\left[\frac{1}{\sqrt{t(1+t)}}\ln(\sqrt{1+t}+\sqrt{t}) - 1\right] \equiv -4g^2 P(t) \quad , \tag{4}$$

$t \equiv -k^2/4m^2,\ \ [g] =$mass. (dim. reg: $D = 4-\epsilon, \epsilon = 2$)

Why in $D=2$ Π is finite?

Taking $k = (0, k_{\|})$, $k^2 = -k_{\|}^2$ for the Coulomb potential in the coordinate representation we get:

$$\mathbf{\Phi}(z) = 4\pi g \int\limits_{-\infty}^{\infty} \frac{e^{ik_{\|}z} dk_{\|}/2\pi}{k_{\|}^2 + 4g^2 P(k_{\|}^2/4m^2)} \quad , \tag{5}$$

and the potential energy for the charges $+g$ and $-g$ is finally: $V(z) = -g\mathbf{\Phi}(z)$.

Rad. corr. to Coulomb potential

Uehling-Serber correction to Coulomb potential.

Who knows?

QED in $D = 4$, no magnetic field
Berestetskii, Lifshitz, Pitaevskii, 4 volume of LL Theor.Phys.

e^2 correction; $exp(-2mr), r >> 1/m$ - very small correction;

logarithmic enhancement of potential (charge growth) for $r << 1/m$ (YM - opposite sign, asymptotic freedom)

Asymptotics of $P(t)$ are:

$$P(t) = \begin{cases} \frac{2}{3}t & , \quad t \ll 1 \\ 1 & , \quad t \gg 1 \quad . \end{cases} \tag{6}$$

Let us take as an interpolating formula for $P(t)$ the following expression:

$$\overline{P}(t) = \frac{2t}{3+2t} \quad . \tag{7}$$

We checked that the accuracy of this approximation is not worse than 10% for the whole interval of t variation, $0 < t < \infty$.

$$\mathbf{\Phi} = 4\pi g \int_{-\infty}^{\infty} \frac{e^{ik_\| z} dk_\| / 2\pi}{k_\|^2 + 4g^2 (k_\|^2/2m^2)/(3 + k_\|^2/2m^2)} =$$

$$= \frac{4\pi g}{1 + 2g^2/3m^2} \int_{-\infty}^{\infty} \left[\frac{1}{k_\|^2} + \frac{2g^2/3m^2}{k_\|^2 + 6m^2 + 4g^2} \right] e^{ik_\| z} \frac{dk_\|}{2\pi} =$$

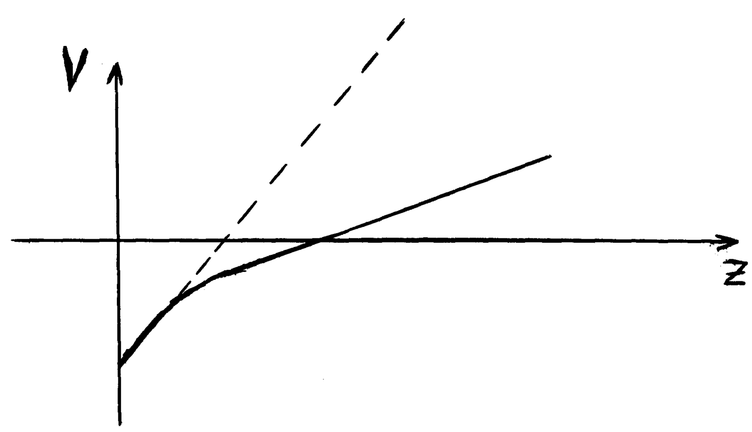
$$= \frac{4\pi g}{1 + 2g^2/3m^2} \left[-\frac{1}{2}|z| + \frac{g^2/3m^2}{\sqrt{6m^2 + 4g^2}} \exp(-\sqrt{6m^2 + 4g^2}|z|) \right]$$

In the case of heavy fermions ($m \gg g$) the potential is given by the tree level expression; the corrections are suppressed as g^2/m^2.

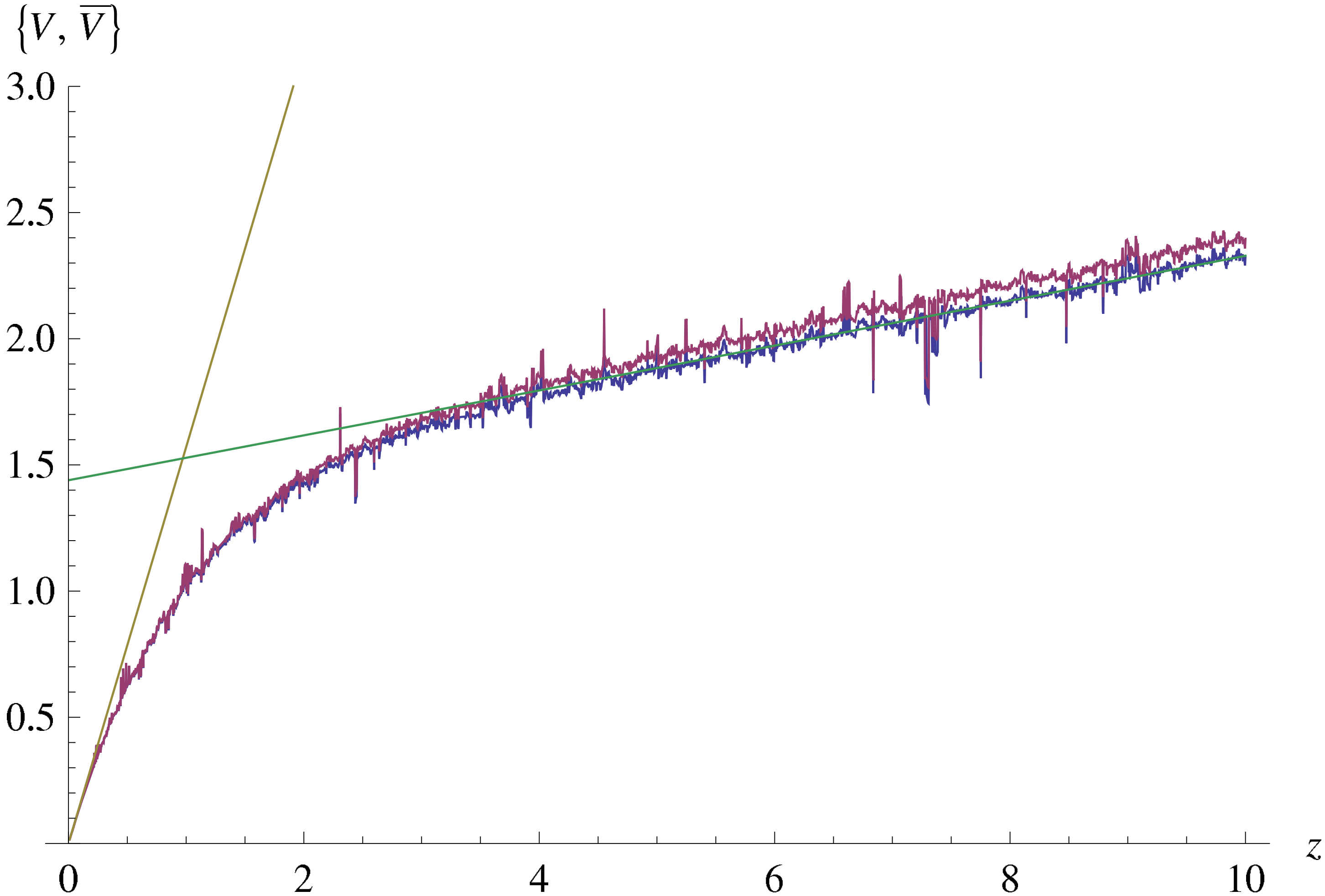
In case of light fermions ($m \ll g$):

$$\mathbf{\Phi}(z)\Big|_{m \ll g} = \begin{cases} \pi e^{-2g|z|} & , \quad z \ll \frac{1}{g} \ln\left(\frac{g}{m}\right) \\ -2\pi g \left(\frac{3m^2}{2g^2}\right)|z| & , \quad z \gg \frac{1}{g} \ln\left(\frac{g}{m}\right) \end{cases} . \tag{9}$$

$m = 0$- Schwinger model; photon get mass. The first gauge invariant theory with massive vector boson (electroweak theory: W, Z). Light fermions:



$$\left\{g = \frac{1}{2},\ m = 0.1\right\}$$



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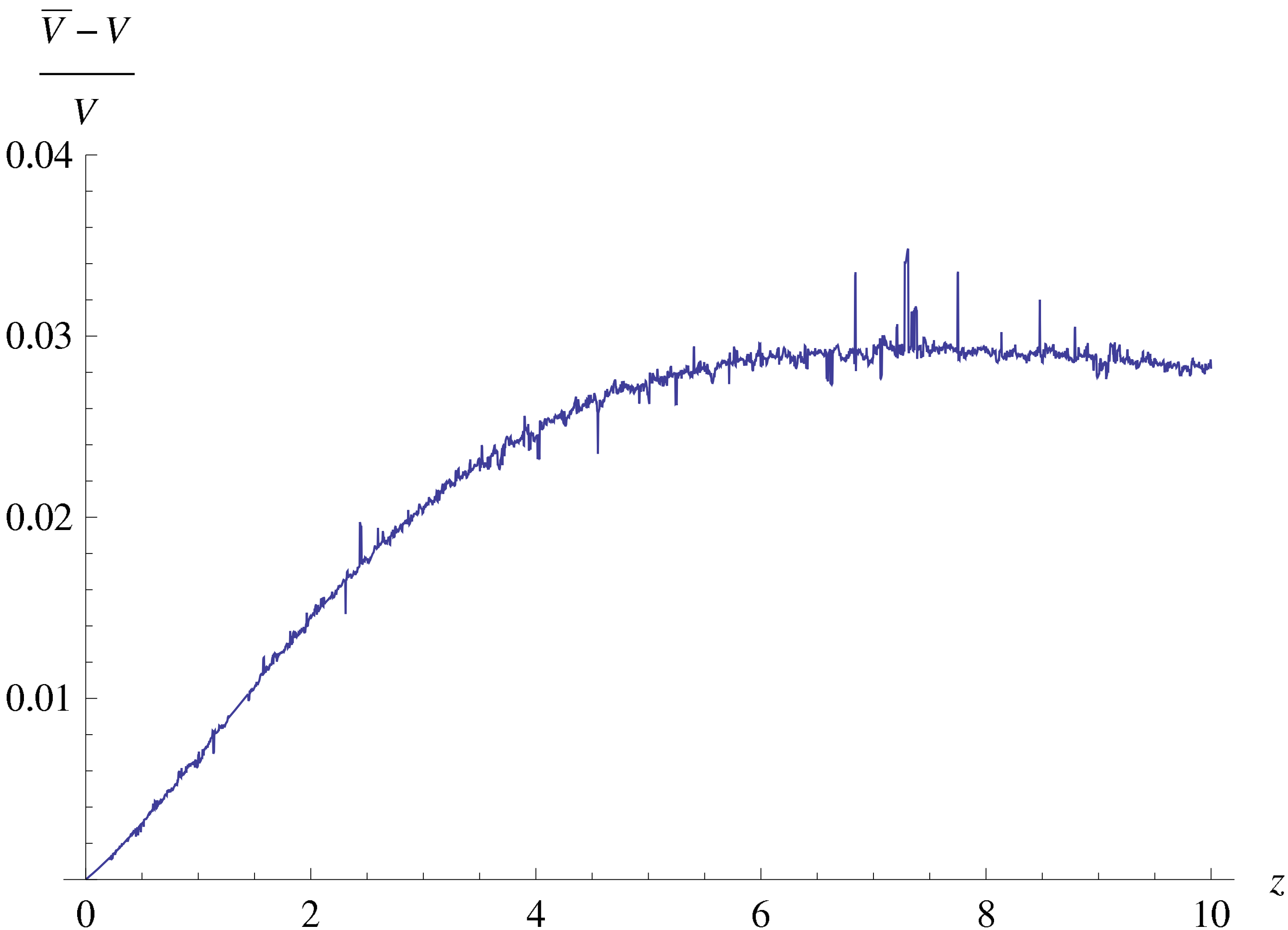

$D = 4$ QED

In order to find potential of pointlike charge we need P in strong B. One starts from electron propagator G in strong B. Solutions of Dirac equation in homogenious constant in time B are known, so one can write spectral representation of electron Green function. Denominators contain $k^2 - m^2 - 2neB$, and for $B >> m^2/e$ and $k_{\|}^2 << eB$ in sum over levels LLL ($n = 0$) dominates. In coordinate representation transverse part of LLL wave function is: $\Psi \sim exp((-x^2 - y^2)eB)$ which in momentum representation gives $\Psi \sim exp((-k_x^2 - k_y^2)/eB)$.

Substituting electron Green functions into polarization operator we get:

$$\Pi_{\mu\nu} \sim e^2 eB \int \frac{dq_x dq_y}{eB} exp(-\frac{q_x^2 + q_y^2}{eB})*$$

$$*exp(-\frac{(q+k)_x^2 + (q+k)_y^2}{eB}) dq_0 dq_z \gamma_\mu \frac{1}{\hat{q}_{0,z} - m} \gamma_\nu \frac{1}{\hat{q}_{0,z} + \hat{k}_{0,z} - m} =$$

$$= e^3 B * exp(-\frac{k_\perp^2}{2eB}) * \Pi_{\mu\nu}^{(2)}(k_\parallel \equiv k_z);$$

$$\mathbf{\Phi} = \frac{4\pi e}{\left(k_\parallel^2 + k_\perp^2\right)\left(1 - \frac{\alpha}{3\pi}\ln\left(\frac{eB}{m^2}\right)\right) + \frac{2e^3 B}{\pi}\exp\left(-\frac{k_\perp^2}{2eB}\right) P\left(\frac{k_\parallel^2}{4m^2}\right)}\,.$$

$$z >> 1/m \Longrightarrow k_{\parallel} << m \Longrightarrow P \sim k_{\parallel}^2/m^2$$

$$\mathbf{\Phi}(z) \sim \int \frac{\exp(ik_{\parallel} z) dk_{\parallel} d^2 k_{\perp}}{k_{\parallel}^2 (1 + e^3 B/m^2) + k_{\perp}^2} \ ,$$

first integrate over $k_{\parallel}$ with the help of residues, after over $k_{\perp}$:

$$\mathbf{\Phi}(z) \Big|_{\ |z| \gg \frac{1}{m}} = \frac{e}{|z|} \, , \ \ V(z) \Big|_{\ z \gg \frac{1}{m}} = -\frac{e^2}{|z|} \qquad (10)$$

$$z << 1/m \Longrightarrow P \sim 1, \mathbf{\Phi} \sim 1/(k_{\|}^2 + k_{\perp}^2 + e^3 B)$$

$$\begin{aligned}
\mathbf{\Phi}(z)\Big|_{\frac{1}{m} \gg z \gg \frac{1}{\sqrt{eB}}} &= e \int_0^{\infty} \frac{\exp\left(-\sqrt{k_{\perp}^2 + \frac{2e^3 B}{\pi}}|z|\right)}{\sqrt{k_{\perp}^2 + \frac{2e^3 B}{\pi}}} k_{\perp} dk_{\perp} = \\
&= \frac{e}{|z|} \exp\left(-\sqrt{\frac{2e^3 B}{\pi}}|z|\right) \quad , \\
V(z) &= -\frac{e^2}{|z|} \exp\left(-\sqrt{\frac{2e^3 B}{\pi}}|z|\right) \quad . \qquad (11)
\end{aligned}$$

atomic levels

Very Preliminary. Equation which gave ground state energy with poor accuracy (but PL, GKK):

$$E_0 = -2m \left(\int_{a_H}^{a_B} U(z) dz \right)^2 \tag{12}$$

(Karnakov, Popov did it much better in 2003 JETP paper). We split the integral into two parts: from $1/m$ to a_B, where the screening is absent (large z),

$$I_1 = - \int_{1/m}^{a_B} \frac{e^2}{z} dz = -e^2 \ln \left(1/e^2\right) \tag{13}$$

and from the Landau radius $a_H = 1/\sqrt{eB}$ to $1/m$, where the screening occurs (small z):

$$I_2 = -\int_{1/\sqrt{eB}}^{1/m} \frac{e^2}{z} \exp(-\sqrt{e^3 B} z) dz = -e^2 \ln(1/e) \quad . \qquad (14)$$

Finally we get:

$$E_0 = -(me^4/2) \times \ln^2(1/e^6) = -(me^4/2) \times 220 \qquad (15)$$

Freezing of ground state energy.
Without screening $I = -e^2 \ln(a_B/a_H)$,
$E_0 = -(me^4/2) \times \ln^2(B/m^2 e^3)$

Shabad, Usov (2007,2008). Analogous consideration to what I told for $D = 4$ + numerical estimates;
$220 \Longrightarrow 295; \;\; 15^2 \Longrightarrow 17^2$

References

Shabad, Usov (2007,2008): $D = 4$ screening of Coulomb potential, freezing of the energy of ground state for $B >> m^2/e^3$;
Skobelev(1975), Loskutov, Skobelev(1976): linear in B term and $D = 4 \Longrightarrow D = 2$ correspondence in photon polarization operator for $B > m^2/e$;
Loskutov, Skobelev(1983); Kuznetsov, Mikheev, Osipov (2002): in $B >> m^2/e^3$ photon “mass” emerge;
Loudon(1959), Elliott, Loudon(1960) - atomic energies in strong $B > m^2e^3$ - numerical calculations;
Karnakov, Popov(2003) - analytical formulas for atomic energies in strong $B > m^2e^3$;

Vysotsky(2010) - analytical formula for ground state hydrogen energy for $B >> m^2/e^3$.

Conclusions

- ground state atomic energy at superstrong B - the only known (for me) case when radiative “correction” determines the energy of state
- analytical expression for charged particle electric potential in $d = 1$ is given; for $m < g$ screening take place at all distances
- asymptotics of potential at superstrong B at $d = 3$ are found
- limit of ground state energy for $B >> m^2/e^3$ is determined analytically: $E_0 = -(me^4/2) \times \ln^2(1/e^6)$